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Non–trivial flat connections on the 3–torus II. The exceptional groups F_4 and $E_{6,7,8}$

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Abstract: We continue the construction of non-trivial vacua for gauge theories on the 3–torus. Application of constructions based on twist in SU(N) with $N > 2$ produce more extra vacua in theories with exceptional groups. We calculate the relevant unbroken subgroups, and their contribution to the Witten index. We show that the extra vacua we find in the exceptional groups are sufficient to solve the Witten index problem for these groups.

Keywords: Gauge Symmetry, Supersymmetry Breaking.

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Contents

1. Introduction

The vacuum equations for Yang-Mills theories on the 3–torus with periodic boundary conditions $F_{\mu\nu} = 0$ always allow for the trivial solution $A_\mu = c_\mu^a H_a$ with c_μ^a a constant and H_a group generators in a maximal abelian algebra of the group (Cartan subalgebra (CSA)). For SU(N) and Sp(N)-theories, all solutions belong to the trivial class, but for other gauge groups this is not the case. In [3, 4] it was shown that other vacua, not belonging to the trivial class, exist for theories with G_2 and SO(N) ($N > 7$) gauge group. In a previous article [5] we proposed a systematic approach to constructing non-trivial vacua in gauge theories on the 3–torus, and applied the method to reproduce the results of [3, 4], and embed these results in the exceptional groups F_4 and $E_{6,7,8}$. In this article we give the details for the extra vacua in theories with exceptional groups, as was announced in [5] (see tables 1 and 2).

The non-trivial vacua of a gauge theory on the 3–torus are of relevance for the problem of computing $\mathrm{Tr}(-1)^F$ in $N = 1$ supersymmetric Yang-Mills theories [1]. As $\mathrm{Tr}(-1)^F$ is an index, assuming no singularities as a function of the volume, this Witten index should be equal in the infinite volume. In the infinite volume the theory has a chiral $U(1)$ symmetry, broken to a discrete $\mathbb{Z}_{2h}$ symmetry by instantons, where h is the dual Coxeter number for the gauge group G. The symmetry is broken further by gluino condensation to a $\mathbb{Z}_2$, leaving h different vacua [1, 2]. In the finite volume version on $T^3 \times R$, the moduli space of flat connections might consist of several components. In each component of the moduli space, the gauge group is partially broken. Performing a Born-Oppenheimer approximation each component of the moduli space, which represents a particular class of solutions, contributes $r' + 1$ to $\mathrm{Tr}(-1)^F$, where r' is the rank of the unbroken gauge group in the relevant component. Equating $\mathrm{Tr}(-1)^F$ for the infinite volume to the result for the finite volume one should find [3]

$$h = \sum_i (r_i + 1) \tag{1.1}$$

where the sum runs over disconnected components of moduli space, i.e. different classes of solutions, and r_i is the rank of the unbroken gauge group in each of these components. This equality is satisfied for theories with SU(N) and Sp(N)-gauge groups [1], where the moduli-space consists of only one component. For the theories with SO(N) groups a second component of the moduli-space was found in [3], and for G_2 the moduli space also consists of two components [4]. In both cases the identity (1.1) is satisfied. For the remaining exceptional groups, a first step towards solving the problem was presented in [5], by showing that the results of [3, 4] can be embedded in these exceptional groups.

Note: after the publication of the first part of this article [5] in which the results of this paper were announced, another paper on the same subject appeared [15]. The results in this paper agree with ours.

2. Method of construction

We will shortly review our method. Instead of giving an explicit expression for the gauge potential, we construct a set of commuting holonomies in a simply connected representation of the gauge group G (in fact, since they commute in a simply-connected representation of the gauge group, they will commute in any representation). A simple theorem [4] then guarantees a corresponding flat connection. To construct commuting elements of a given group, we study a PG'_N-subgroup, whose universal covering $\widetilde{PG'}_N$ is a product of N factors $\tilde{G}'$. The global structure of PG'_N is not that of a product group, since there is a diagonal central subgroup divided out. This allows us to impose so-called multi-twisted boundary conditions [7]. As an example we consider a subgroup PG'_2 (with universal covering $\tilde{G}'^2$). We assume that $\tilde{G}'$ allows a non-trivial centre Z, and that the global subgroup PG'_2 is a representation of $(\tilde{G}' \times \tilde{G}')/Z_{diag}$ where $Z_{diag} \cong Z$ is a diagonal subgroup of the centre $Z \times Z$ of $\tilde{G}'^2$. We select two elements P_1, Q_1 generated by the Lie algebra of the first $\tilde{G}'$ factor $(\tilde{G}'_1)$, and two elements P_2, Q_2 generated by the Lie algebra of the second $\tilde{G}'$ factor $(\tilde{G}'_2)$, such that they commute up to a non-trivial element of the centre Z of $\tilde{G}'_i$ [6]:

$$P_1 Q_1 = z_1 Q_1 P_1 \tag{2.1}$$

$$P_2 Q_2 = z_2 Q_2 P_2 \,. \tag{2.2}$$

Since we only deal with representations of $(\tilde{G}' \times \tilde{G}')/Z_{diag}$, z_1 and z_2 are actually elements of the central subgroup $(Z \times Z)/Z_{diag} \cong Z$. By picking P_i and Q_i in a specific way, we can thus arrange that (2.1), (2.2) are satisfied with the additional condition

$$z_1 = (z_2)^{-1} \equiv z \,. \tag{2.3}$$

We now construct the commuting elements

$$P = P_1 P_2 \,, \qquad Q = Q_1 Q_2 \,, \qquad P' = Q_1^n P Q_1^{-n} \,. \tag{2.4}$$

These elements will serve as candidate holonomies. To make sure that one finds a non-trivial vacuum, one has to calculate the subgroup left unbroken by the holonomies, and verify that its rank is lower than that of the original gauge group. For a more thorough discussion, we refer the reader to [5], in that paper we discussed results based on subgroups PG'_M with $\widetilde{PG'}_M = \mathrm{SU}(2)^M$. We now proceed with results based on subgroups PG'_M with $\widetilde{PG'}_M = \mathrm{SU}(N)^M$, for $N > 2$.

3. Representations of $\mathrm{SU}(N > 2)^M$

Before starting the discussion on explicit realisations of our construction we wish to point out a few subtleties about these groups.

The groups $\mathrm{SU}(N > 2)$ posses an outer automorphism that corresponds to complex conjugation. In other words, unlike the situation for SU(2), an $\mathrm{SU}(N > 2)$

irreducible representation (irrep) is in general not equivalent to the complex conjugate of the irrep. The complex conjugated irrep has the same dimension as the original irrep, and in the notation where each irrep is denoted by its dimension, say $\mathbf{n}$, the complex conjugate irrep is denoted by the dimension with a bar on top $\bar{\mathbf{n}}$. It is a matter of convention which irrep is labelled as $\mathbf{n}$ and which as $\bar{\mathbf{n}}$. In the root diagram of the corresponding algebra, this corresponds to an ambiguity in labelling the roots. We will be interested in product groups, where one has this ambiguity in every factor. The choice we will make to label the representations is related to the following issue.

A diagonal subgroup of a group PG'_N can be constructed as follows: construct a Lie algebra $\mathcal{L}$ for $\tilde{G}'$, consisting of elements T^a. Then the Lie algebra for PG'_N has the structure $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_N$ with each of the $\mathcal{L}_i \cong \mathcal{L}$. Hence we can write T^a_i, for the generator from $\mathcal{L}_i$ that corresponds to T^a under an isomorphism mapping $\mathcal{L}$ to $\mathcal{L}_i$. The diagonal subgroup of PG'_N is then constructed by taking as generators $T^a_1 + T^a_2 + \cdots + T^a_N$. This construction is not unique, there are many isomorphisms from $\mathcal{L}_i$ to $\mathcal{L}$, and these will give different diagonal subgroups. All these diagonal subgroups have the same Lie algebra, but they can have a different global structure. As an example consider the $\mathrm{SU}(3) \times \mathrm{SU}(3)$ irrep $(\mathbf{3}, \bar{\mathbf{3}})$, which in a different labelling would be denoted as $(\mathbf{3}, \mathbf{3})$. Constructing the diagonal subgroup of $(\mathbf{3}, \bar{\mathbf{3}})$ as outlined in the above, will lead to a tensor representation which decomposes into an $\mathbf{8}$ and a $\mathbf{1}$. The diagonal subgroup of $(\mathbf{3}, \mathbf{3})$ however is a tensor representation which decomposes in a $\mathbf{6}$ and a $\bar{\mathbf{3}}$. We will prefer the labelling $(\mathbf{3}, \bar{\mathbf{3}})$ since then the diagonal subgroup (which is actually $\mathrm{SU}(3)/\mathbb{Z}_3$) will have a trivial centre.

We construct our holonomies in subgroups that are not simply connected, regardless of the representation of G we are working in. We need to verify case by case that the holonomies that we construct commute in a simply connected representation, implying that they commute in any representation of the gauge group G.

For each algebra, there exists a set of "fundamental weights" Λ_i, defined by

$$2\frac{\langle \Lambda_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}\,, \tag{3.1}$$

where the α_j are the simple roots of the algebra. Λ_i can be written as a linear combination of simple roots with rational coefficients. Each weight is of the form

$$\lambda = \sum_i (n_i \Lambda_i + m_i \alpha_i) \quad \text{with } n_i, m_i \in \mathbb{Z}\,. \tag{3.2}$$

When constructing a subalgebra $\mathcal{L}_{G'}$ of an algebra $\mathcal{L}_G$, one can arrange that the CSA of $\mathcal{L}_{G'}$ is contained in the CSA of $\mathcal{L}_G$. Call the generators of $\mathcal{L}_{G'}$, $H_{\beta'}$ and $E_{\beta'}$. Then we have $H_{\beta'} = h_\beta$ with $h_\beta \in \mathcal{L}_G$. To find the weights of the subgroup, we first find the h_β associated to simple roots of G'. These can be calculated using:

$$h_\beta = H_{\beta'} = \frac{[E_{\beta'}, E_{-\beta'}]}{2}\,. \tag{3.3}$$

The weights λ' of G' can now be found from the weights λ of G with

$$\langle \lambda', \beta' \rangle = \langle \lambda, \beta \rangle \,. \tag{3.4}$$

One way to find out what the congruence class of the representation of G' is, is by expressing the weights λ' of the representation of G' in the simple roots α'_j of G'. We write

$$\lambda' = \sum_k q_k \alpha'_k$$

The q_k are easily calculated using (3.1) with Λ'_k a fundamental weight of G'

$$2\langle \Lambda'_k, \lambda' \rangle = q_k \langle \alpha'_k, \alpha'_k \rangle \,. \tag{3.5}$$

In all cases we will be considering $G' = \mathrm{SU}(N)$, and hence $\langle \alpha'_k, \alpha'_k \rangle$ is a k-independent number. Since all weights can be obtained from one weight by adding (or subtracting) simple roots, the fractional part of the q_k is sufficient to determine the weight lattice, and thereby the congruence class of the representation.

We will now continue with the explicit constructions. Our conventions are outlined in the appendices of [5]. For the decompositions of groups into subgroups, use was made of [9].

4. Non-trivial vacua based on twist in SU(3)

We will start by developing the relevant tools for this group. After that we will give an overview of groups in which our construction can be realised. These are all the exceptional groups except G_2. F_4 will be discussed in quite some detail.

Like in [5] where $\tilde{G}' = \mathrm{SU}(2)$ was discussed, our construction can always be carried out in a subgroup $\mathrm{SU}(3)^2$, but we will often take $\mathrm{SU}(3)^M$ with $M > 2$, as this allows us to choose it to be a regular subgroup. This will give important simplifications in the calculations.

4.1 Twist in SU(3)

We will take the canonical form (in the conventions of the appendices of [5], with the modification that we will use capital E and H to be able to distinguish the subgroup generated by these from the original group, whose generators we keep on denoting by h_α and e_α):

$$\begin{aligned}
&[H_{\alpha_1}\,,\,E_{\alpha_1}] = \frac{1}{3}E_{\alpha_1} \qquad &&[H_{\alpha_1}, E_{\alpha_2}] = -\frac{1}{6}E_{\alpha_1} \qquad &&[H_{\alpha_1}, E_{\alpha_1+\alpha_2}] = \frac{1}{6}E_{\alpha_1+\alpha_2}\\
&[H_{\alpha_2}\,,\,E_{\alpha_1}] = -\frac{1}{6}E_{\alpha_1} &&[H_{\alpha_2}, E_{\alpha_2}] = \frac{1}{3}E_{\alpha_2} &&[H_{\alpha_2}, E_{\alpha_1+\alpha_2}] = \frac{1}{6}E_{\alpha_1+\alpha_2}\\
&[E_{\alpha_1}, E_{-\alpha_1}] = 2H_{\alpha_1} &&[E_{\alpha_2}, E_{-\alpha_2}] = 2H_{\alpha_2} &&[E_{\alpha_1}, E_{\alpha_2}] = \frac{1}{\sqrt{3}}E_{\alpha_1+\alpha_2}\\
&[E_{\alpha_1+\alpha_2}, E_{-\alpha_1-\alpha_2}] = 2H_{\alpha_1+\alpha_2} = 2H_{\alpha_1} + 2H_{\alpha_2}
\end{aligned} \tag{4.1}$$

All other relations can be found by conjugation, and the Jacobi identity.

In SU(3) we look for two matrices satisfying

$$pq = \exp\left(\frac{2\pi i}{3}\right) qp \tag{4.2}$$

p and q will commute when lifted to $\mathrm{SU}(3)/\mathbb{Z}_3$. We take:

$$p = \begin{pmatrix} \exp(\frac{2\pi i}{3}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(-\frac{2\pi i}{3}) \end{pmatrix}, \qquad q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{4.3}$$

In terms of generators this is

$$\begin{aligned} p &= \exp\left(4\pi i H_{\alpha_1+\alpha_2}\right) \\ q &= \exp\left(\frac{4\pi i}{3}\frac{1}{2i}(E_{\alpha_1} + E_{\alpha_2} + E_{-\alpha_1-\alpha_2} - E_{-\alpha_1} - E_{-\alpha_2} - E_{\alpha_1+\alpha_2})\right) \end{aligned} \tag{4.4}$$

The commutation relations of p and q with the group generators are most easily calculated in a specific representation. One finds

$$\begin{aligned} pH_\alpha &= H_\alpha p, & qH_\alpha &= H_{R\alpha} q, \\ pE_\alpha &= \exp(4\pi i\langle\alpha, \alpha_1+\alpha_2\rangle)E_\alpha p, & qE_\alpha &= E_{R\alpha} q\,. \end{aligned} \tag{4.5}$$

The action of the rotation R (this is a genuine rotation over $2\pi/3$ in the root diagram) is fully determined by its action on the simple roots. R is an element of the Weyl group: It is the composition of the Weyl reflection generated by α_2, followed by the reflection generated by α_1. The effect of R is

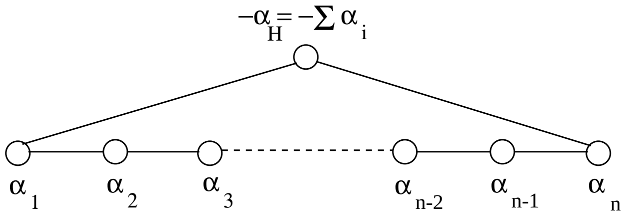

Figure 1: The extended Dynkin diagram for $\mathrm{SU}(N+1)$

$$R: \qquad \alpha_1 \to \alpha_2 \to -(\alpha_1+\alpha_2) \to \alpha_1 \tag{4.6}$$

A nice mnemonic for the action of R is provided by the "extended Dynkin diagram" [8] of SU(3). The extended Dynkin diagram of an algebra consists of its Dynkin diagram, extended with one more root, $-\alpha_H$, with α_H the highest root of the algebra. The dot representing the highest root is then connected to the diagram via the standard rules. For su(3), $-\alpha_H = -(\alpha_1+\alpha_2)$ and the extended diagram consists of 3 dots, connected to form a cycle. The action of R is nothing but a rotation of this cycle by one step. For twist in the higher unitary groups we will see the same feature: the action of the analogon of the element q can be represented by rotating the (cyclic) extended Dynkin diagram of the unitary group by one step.

4.2 Calculation of the unbroken subgroup

Again we wish to compute the combinations $pT^a p^{-1}$ and $qT^a q^{-1}$, with T^a the generators of G. We will generalise our methods for the SU(2)-case to SU(3), but this is complicated by a crucial difference between SU(2) and SU(3): For SU(2) all weights in an irreducible representation have multiplicity 1, whereas this is not true for SU(3)-irreps. However, we will need only three irreps of SU(3): the **1** and **10**, which do contain only simple weights, and the **8**, which is equivalent to the adjoint. Note that these are all irreps of SU(3)/$\mathbb{Z}_3$.

We proceed as in the SU(2)-case. $pT^a p^{-1}$ is easily calculated, with $p = \exp(ih_\zeta)$ we have

$$ph_\alpha p^{-1} = h_\alpha \,, \qquad pe_\beta p^{-1} = \exp(i\langle\zeta,\beta\rangle)e_\beta \,. \tag{4.7}$$

To calculate $qT^a q^{-1}$ we start by decomposing the representation of the group G into irreducible representations of SU(3)/$\mathbf{Z_3}$. First we construct the normalised eigenvectors ψ_λ of the CSA: $H_\alpha\psi_\lambda = \langle\alpha,\lambda\rangle\psi_\lambda$. It then follows that

$$H_{R\alpha}q\psi_\lambda = qH_\alpha\psi_\lambda = \langle\alpha,\lambda\rangle q\psi_\lambda = \langle R\alpha, R\lambda\rangle q\psi_\lambda \,. \tag{4.8}$$

This means that $q\psi_\lambda$ is an eigenvector of the CSA with weight $R\lambda$. If all weights are simple, as in the **1** and **10**, then everything is easily solvable, as in the SU(2) case: $q\psi_\lambda = \phi_\lambda\psi_{R\lambda}$ with ϕ_λ a phase. From $q^3 = 1$, we find that $\phi_\lambda\phi_{R\lambda}\phi_{R^2\lambda} = 1$. With the ladder operators one shows that ϕ_λ is actually independent of λ (in a suitable phase convention). The following three combinations are eigenvectors:

$$\begin{aligned}
\psi_\lambda + \psi_{R\lambda} + \psi_{R^2\lambda} &\rightarrow \text{eigenvalue } \phi \\
\psi_\lambda + \exp\left(\frac{4\pi i}{3}\right)\psi_{R\lambda} + \exp\left(\frac{2\pi i}{3}\right)\psi_{R^2\lambda} &\rightarrow \text{eigenvalue } \exp\left(\frac{2\pi i}{3}\right)\phi \\
\psi_\lambda + \exp\left(\frac{2\pi i}{3}\right)\psi_{R\lambda} + \exp\left(\frac{4\pi i}{3}\right)\psi_{R^2\lambda} &\rightarrow \text{eigenvalue } \exp\left(\frac{4\pi i}{3}\right)\phi
\end{aligned}$$

Note that formally we have to turn to the complexification of the real Lie-algebra for this to make sense. For $\lambda = 0$ the last two of these combinations are zero. We find that each triple of eigenvectors contributes $\phi^3 = 1$ to the determinant, and since all weights, apart from the zero weight, occur in triples we find that the determinant is equal to $\phi^{3n+1} = \phi = 1$.

When not all weights have multiplicity 1, the above discussion for the non-zero weights applies anyway. It is the zero weight that causes the problems. The eigenvectors with weight zero form a subspace that will be mapped into itself, our previous methods fail, and we know no easy way to determine the action of q on any vector of this subspace. For the **8**, the solution is nevertheless easy to find by realising that this is the adjoint. The zero weights of the adjoint representation are the generators of the CSA, the H_α, on which we already know the action of q, and the eigenvectors of q are

$$
\begin{aligned}
H_\alpha + H_{R\alpha} + H_{R^2\alpha} &= 0 \\
H_\alpha + \exp\left(\frac{4\pi i}{3}\right) H_{R\alpha} + \exp\left(\frac{2\pi i}{3}\right) H_{R^2\alpha} &\to \text{eigenvalue } \exp\left(\frac{2\pi i}{3}\right) \\
H_\alpha + \exp\left(\frac{2\pi i}{3}\right) H_{R\alpha} + \exp\left(\frac{4\pi i}{3}\right) H_{R^2\alpha} &\to \text{eigenvalue } \exp\left(\frac{4\pi i}{3}\right)
\end{aligned}
$$

We can now construct the action of q on any representation of G that splits into singlets, octets and decuplets. As a final remark we note that these considerations imply that

$$
pT^a p^{-1} = \exp\left(\frac{2\pi i n}{3}\right) T^a\,, \qquad qT^a q^{-1} = \exp\left(\frac{2\pi i m}{3}\right) T^a\,.
$$

For the su(3)-subalgebra, we define the CSA generators $H_{\alpha'_1}$ and $H_{\alpha'_2}$ associated to the simple roots of su(3), following the conventions of the appendix of [5]. Set $H_{\alpha'_1} = h_{\beta_1}$, $H_{\alpha'_2} = h_{\beta_2}$ for $h_{\beta_i} \in \mathcal{L}_G$. We need the combinations $2\Lambda'_i/\langle\alpha'_k, \alpha'_k\rangle$ (see eq. (3.5)), where Λ'_i are the fundamental weights of su(3) and α'_k are simple roots of su(3). In our conventions $\langle\alpha'_k, \alpha'_k\rangle = 1/3$, for α'_k a root of su(3). The combinations we need are thus

$$
6\Lambda'_1 = 4\beta_1 + 2\beta_2\,, \qquad 6\Lambda'_2 = 2\beta_1 + 4\beta_2\,.
$$

We write the weights of the representation of the group G as (see (3.2))

$$
\lambda = \sum_i (n_i\Lambda_i + m_i\alpha_i)\,, \qquad \text{with } n_i, m_i \in \mathbb{Z}\,,
$$

with Λ_i the fundamental weights of G and α_i its simple roots. We will demand that the representation of G branches into representations of $\mathrm{SU}(3)/\mathbb{Z}_3$, a condition that can be translated to

$$
\langle\alpha_i, 4\beta_1 + 2\beta_2\rangle\,, \quad \langle\alpha_i, 2\beta_1 + 4\beta_2\rangle\,, \quad \langle\Lambda_i, 4\beta_1 + 2\beta_2\rangle\,, \quad \langle\Lambda_i, 2\beta_1 + 4\beta_2\rangle \quad \in \mathbb{Z} \tag{4.9}
$$

4.3 Realisation of the SU(3)-based construction

4.3.1 F_4

F_4 possesses a PG'_2-subgroup with $\widetilde{PG'}_2 = \mathrm{SU}(3)^2$. The roots of the subalgebra are contained in the f_4 root lattice. An su(3)-sublattice is completely determined by making a choice for its simple roots. We take for the first su(3) factor the two roots $(1,-1,0,0)/\sqrt{18}$ and $(0,1,-1,0)/\sqrt{18}$, which are of equal length and have an angle in-between of $2\pi/3$ and can therefore be used as simple roots for su(3). For the second su(3) factor we take as simple roots $(0,0,0,1)/\sqrt{18}$ and $(1,1,1,-1)/2\sqrt{18}$.

JHEP05(1999)014

These two are orthogonal to the root vectors of the first su(3). To construct the su(3) algebra's, we need to include appropriate normalisation factors. The first SU(3)-factor is generated by:

$$
\begin{aligned}
&H^1_{\alpha_1} = 3h_{(1,-1,0,0)/\sqrt{18}}\,, \qquad H^1_{\alpha_2} = 3h_{(0,1,-1,0)/\sqrt{18}} \\
&E^1_{\alpha_1} = \sqrt{3}e_{(1,-1,0,0)/\sqrt{18}}\,, \qquad E^1_{\alpha_2} = \sqrt{3}e_{(0,1,-1,0)/\sqrt{18}}\,, \qquad E^1_{\alpha_1+\alpha_2} = \sqrt{3}e_{(1,0,-1,0)/\sqrt{18}}\,.
\end{aligned}
\tag{4.10}
$$

The second SU(3) factor is generated by

$$
\begin{aligned}
&H^1_{\alpha_1} = 6h_{(0,0,0,1)/\sqrt{18}} \qquad H^1_{\alpha_2} = 6h_{(1,1,1,-1)/2\sqrt{18}} \\
&E^1_{\alpha_1} = \sqrt{6}e_{(0,0,0,1)/\sqrt{18}} \qquad E^1_{\alpha_2} = \sqrt{6}e_{(1,1,1,-1)/2\sqrt{18}} \qquad E^1_{\alpha_1+\alpha_2} = \sqrt{6}e_{(1,1,1,1)/2\sqrt{18}}\,.
\end{aligned}
\tag{4.11}
$$

The diagonal subalgebra D is then easily constructed:

$$
\begin{aligned}
H^D_{\alpha_1} &= H^1_{\alpha_1} + H^2_{\alpha_1} &&= \frac{1}{2}\sqrt{2}h_{(1,-1,0,2)} \\
H^D_{\alpha_2} &= H^1_{\alpha_2} + H^2_{\alpha_2} &&= \frac{1}{2}\sqrt{2}h_{(1,2,0,-1)} \\
E^D_{\alpha_1} &= E^1_{\alpha_1} + E^2_{\alpha_1} &&= \sqrt{3}e_{(1,-1,0,0)/\sqrt{18}} + \sqrt{6}e_{(0,0,0,1)/\sqrt{18}} \\
E^D_{\alpha_2} &= E^1_{\alpha_2} + E^2_{\alpha_2} &&= \sqrt{3}e_{(0,1,-1,0)/\sqrt{18}} + \sqrt{6}e_{(1,1,1,-1)/2\sqrt{18}} \\
E^D_{\alpha_1+\alpha_2} &= E^1_{\alpha_1+\alpha_2} + E^2_{\alpha_1+\alpha_2} &&= \sqrt{3}e_{(1,0,-1,0)/\sqrt{18}} + \sqrt{6}e_{(1,1,1,1)/2\sqrt{18}}\,.
\end{aligned}
\tag{4.12}
$$

This algebra generates an $\mathrm{SU}(3)/\mathbb{Z}_3$-group, as can be seen by checking the conditions (4.9). We have $4\alpha_1 + 2\alpha_2 = \sqrt{18}(1,0,0,1)$ and $2\alpha_1 + 4\alpha_2 = \sqrt{18}(1,1,0,0)$, and the inner product of these vectors with the simple roots of F_4 all give integers. F_4 has no non-integer fundamental weights. The decomposition of F_4 into the diagonal $\mathrm{SU}(3)/\mathbb{Z}_3$ is as follows:

$$
\begin{array}{llll}
F_4 & \to & \mathrm{SU}(3)\times\mathrm{SU}(3) & \to \ \mathrm{SU}(3)/\mathbb{Z}_3 \\
\mathbf{26} & \to & (\mathbf{3},\bar{\mathbf{3}}) \oplus (\bar{\mathbf{3}},\mathbf{3}) \oplus (\mathbf{8},\mathbf{1}) & \to \ 3(\mathbf{8}) \oplus 2(\mathbf{1}) \\
\mathbf{52} & \to & (\mathbf{8},\mathbf{1}) \oplus (\mathbf{6},\mathbf{3}) \oplus (\bar{\mathbf{6}},\bar{\mathbf{3}}) \oplus (\mathbf{1},\mathbf{8}) & \to \ (\mathbf{10}) \oplus (\overline{\mathbf{10}}) \oplus 4(\mathbf{8})\,.
\end{array}
\tag{4.13}
$$

Now look for two matrices that commute, but cannot be written as exponentials of generators of D that commute. From (4.4)

$$
\begin{aligned}
P &= \exp\left(4\pi i H^D_{\alpha_1+\alpha_2}\right) \\
Q &= \exp\left(\frac{4\pi i}{3}\frac{1}{2i}(E^D_{\alpha_1} + E^D_{\alpha_2} + E^D_{-\alpha_1-\alpha_2} - E^D_{-\alpha_1} - E^D_{-\alpha_2} - E^D_{\alpha_1+\alpha_2})\right).
\end{aligned}
\tag{4.14}
$$

Splitting the generators of the diagonal group as $H^D_\alpha = H^1_\alpha + H^2_\alpha$ and $E^D_\alpha = E^1_\alpha + E^2_\alpha$ reveals the product structure of P and Q, which decompose accordingly as $P = P_1P_2$, and $Q = Q_1Q_2$. That P and Q commute follows from the fact that they take the role of the elements p and q constructed previously, and that they are elements of an

$SU(3)/\mathbb{Z}_3$-subgroup. Now we wish to construct P'. Our methods for SU(2) [5] have to be generalised in such a way, that it produces a subgroup D' that is isomorphic to D, such that Q is an element of both D and D'. The appropriate generalisation has the form of a rotation of one of the two factors

$$H^1_\alpha \to H^1_{R\alpha} \qquad E^1_\alpha \to E^1_{R\alpha}\,, \tag{4.15}$$

where R is the rotation (4.6). The second SU(3) will be left as it is. We now construct the diagonal group D':

$$\begin{aligned}
H^{D'}_{\alpha_1} &= \quad H^1_{\alpha_2} + H^2_{\alpha_1} &&= \frac{1}{2}\sqrt{2}h_{(0,1,-1,2)}\\
H^{D'}_{\alpha_2} &= H^1_{-\alpha_1-\alpha_2} + H^2_{\alpha_2} &&= \frac{1}{2}\sqrt{2}h_{(0,1,2,-1)}\\
E^{D'}_{\alpha_1} &= \quad E^1_{\alpha_2} + E^2_{\alpha_1} &&= \sqrt{3}e_{(0,1,-1,0)/\sqrt{18}} + \sqrt{6}e_{(0,0,0,1)/\sqrt{18}}\\
E^{D'}_{\alpha_2} &= E^1_{-\alpha_1-\alpha_2} + E^2_{\alpha_2} &&= \sqrt{3}e_{(-1,0,1,0)/\sqrt{18}} + \sqrt{6}e_{(1,1,1,-1)/2\sqrt{18}}\\
E^{D'}_{\alpha_1+\alpha_2} &= E^1_{-\alpha_1} + E^2_{\alpha_1+\alpha_2} &&= \sqrt{3}e_{(-1,1,0,0)/\sqrt{18}} + \sqrt{6}e_{(1,1,1,1)/2\sqrt{18}}\,.
\end{aligned} \tag{4.16}$$

We set P' to be

$$P' = \exp(4\pi i H^{D'}_{\alpha_1+\alpha_2})\,, \tag{4.17}$$

P, P' and Q commute by construction. Since F_4 is simply connected, the gauge connection implied by the holonomies $\Omega_1 = P$, $\Omega_2 = P'$ and $\Omega_3 = Q$ is flat and non-trivial.

Now we calculate the unbroken subgroup: first find the generators that commute with P, of these check how many commute with P', and finally compute the commutator with Q of the generators that commute with both P and P'. These computations can be done by using our previous results and substituting P for p and Q for q in the first step, and in a second step P' for p and Q for q . The only $e_{\pm\beta}$ that commute with P have

$$\beta \in \left\{ \frac{(0,0,1,0)}{\sqrt{18}}, \frac{(1,1,0,0)}{\sqrt{18}}, \frac{(1,0,0,1)}{\sqrt{18}}, \frac{(0,1,0,-1)}{\sqrt{18}}, \frac{(1,-1,1,-1)}{2\sqrt{18}}, \frac{(1,-1,-1,-1)}{2\sqrt{18}} \right\}$$

However, none of these commute with P'. Hence the CSA-generators are the only ones that commute with both P and P'. The effect of Q on the CSA can be computed by studying the branching of F_4 into $SU(3)/\mathbb{Z}_3$. Careful examination shows that all CSA-generators have weight zero for the **8**'s in the decomposition. There is no combination of zero weights for the **8** that is invariant under the adjoint action of Q and hence no generator commutes with P, P' and Q, as will be proven in a subsequent publication. The unbroken subgroup is therefore discrete.

We therefore have found a *new* flat connection: Neither the trivial flat connection nor the flat connection constructed in [5] can be deformed to a flat connection that has a discrete unbroken subgroup. Note that instead of the rotation (4.15) we could also have rotated by

$$H^1_\alpha \to H^1_{R^2\alpha}\,, \qquad E^1_\alpha \to E^1_{R^2\alpha}\,, \tag{4.18}$$

Repeating the steps for the construction of D', this gives another diagonal subgroup, that we will call D''. It is also possible to construct an element $P'' = \exp(4\pi i H^{D''}_{\alpha_1+\alpha_2})$. P'' commutes with P, P', and Q, and can be used in the construction of non-trivial flat connections. The flat connection corresponding to $\Omega_1 = P$, $\Omega_2 = P'$ and $\Omega_3 = Q$ is not equivalent to the one corresponding to $\Omega_1 = P$, $\Omega_2 = P''$ and $\Omega_3 = Q$. These two flat connections both have discrete unbroken subgroups.

We remind the reader that the non-trivial flat connection in SO(7) in [3] was characterised by simultaneously diagonalising the three holonomies in the 7 dimensional vector representation. The 7 triples $((\Omega_1)_{ii}, (\Omega_2)_{ii}, (\Omega_3)_{ii})$ are the triples $(\pm1, \pm1, \pm1)$ with at least one -1. In Witten's original construction, the triples $(\pm1, \pm1, \pm1)$ correspond to the positions of D-branes at orientifold fixed points.

There is a remarkable resemblance between this result and the SU(3)-based non-trivial flat connection in F_4. We can diagonalise the holonomies constructed here. On the diagonal we have the eigenvalues $\exp(2\pi i n/3)$. Take now the triples

$$\left(\exp\left(\frac{2\pi i n_1}{3}\right), \exp\left(\frac{2\pi i n_2}{3}\right), \exp\left(\frac{2\pi i n_3}{3}\right)\right), \qquad n_i \in \mathbb{Z} \tag{4.19}$$

and exclude the triple $(1, 1, 1)$. There are $3^3 - 1 = 26$ distinct triples, and this is precisely the dimension of the fundamental irrep of F_4. Indeed, constructing the elements P, P' and Q in the **26** of F_4 and diagonalising, we find that the triples of diagonal elements $(P_{ii}, P'_{ii}, Q_{ii})$ are precisely the 26 triples mentioned above. It is an intriguing question whether this too can be related to an M-theory construction.

4.3.2 E_6

In E_6 there is a $\mathrm{SU}(3)^3$-subgroup that is suitable for our purposes. The su(3)-factors have root vectors

$$\begin{aligned}
\mathrm{su}(3)_1 : \alpha^1_1 &= \frac{(0,0,0,0,1,1)}{\sqrt{24}}, & \alpha^1_2 &= \frac{(\sqrt{3},-1,-1,-1,-1,-1)}{2\sqrt{24}}\,;\\
\mathrm{su}(3)_2 : \alpha^2_1 &= \frac{(0,0,1,-1,0,0)}{\sqrt{24}}\,, & \alpha^2_2 &= \frac{(0,1,-1,0,0,0)}{\sqrt{24}}\,;\\
\mathrm{su}(3)_3 : \alpha^3_1 &= \frac{(0,0,0,0,1,-1)}{\sqrt{24}}\,, & \alpha^3_2 &= \frac{(-\sqrt{3},-1,-1,-1,-1,1)}{2\sqrt{24}}\,.
\end{aligned} \tag{4.20}$$

The relevant geometrical properties are easily verified. To construct the su(3) algebra's, we need to include appropriate normalisation factors. A full exposition would be very space consuming, so we limit ourselves to a few points.

The diagonal subalgebra D is generated by:

$$\begin{aligned}
H^D_{\alpha_1} &= \quad H^1_{\alpha_1}+H^2_{\alpha_1}+H^3_{\alpha_1} &&= \frac{1}{3}\sqrt{6}h_{(0,0,1,-1,2,0)}\\
H^D_{\alpha_2} &= \quad H^1_{\alpha_2}+H^2_{\alpha_2}+H^3_{\alpha_1} &&= \frac{1}{3}\sqrt{6}h_{(0,0,-2,-1,-1,0)}\\
E^D_{\alpha_1} &= \quad E^1_{\alpha_1}+E^2_{\alpha_1}+E^3_{\alpha_1} &&= 2(e_{\alpha_1^1}+e_{\alpha_1^2}+e_{\alpha_1^3})\\
E^D_{\alpha_2} &= \quad E^1_{\alpha_2}+E^2_{\alpha_2}+E^3_{\alpha_2} &&= 2(e_{\alpha_2^1}+e_{\alpha_2^2}+e_{\alpha_2^3})\\
E^D_{\alpha_1+\alpha_2} &= E^1_{\alpha_1+\alpha_2}+E^2_{\alpha_1+\alpha_2}+E^3_{\alpha_1+\alpha_2} &&= 2(e_{\alpha_1^1+\alpha_2^1}+e_{\alpha_1^2+\alpha_2^2}+e_{\alpha_1^3+\alpha_2^3})
\end{aligned} \tag{4.21}$$

This algebra generates an SU(3)/$\mathbb{Z}_3$-group, as can be seen by checking the conditions (4.9). This subgroup corresponds to the decompositions

$$\begin{array}{lllll}
E_6 & \to & \mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{SU}(3) & \to & \mathrm{SU}(3)/\mathbb{Z}_3\\
\mathbf{27} & \to & (\mathbf{3},\bar{\mathbf{3}},\mathbf{1})\oplus(\bar{\mathbf{3}},\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\mathbf{3},\bar{\mathbf{3}}) & \to & 3(\mathbf{8})\oplus 3(\mathbf{1})\\
\mathbf{78} & \to & (\mathbf{8},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{8},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{8}) & & \\
 & & \oplus(\mathbf{3},\mathbf{3},\mathbf{3})\oplus(\bar{\mathbf{3}},\bar{\mathbf{3}},\bar{\mathbf{3}}) & \to & (\mathbf{10})\oplus(\overline{\mathbf{10}})\oplus 7(\mathbf{8})\oplus 2(\mathbf{1})\,.
\end{array} \tag{4.22}$$

The diagonal subalgebra's D' and D'' are obtained by rotating one of the three SU(3) factors by (4.15) resp. (4.18). The holonomies are then $\Omega_1 = P$, $\Omega_2 = P'$ and $\Omega_3 = Q$ with

$$\begin{aligned}
P &= \exp(4\pi i H^D_{\alpha_1+\alpha_2})\,, \qquad P' = \exp(4\pi i H^{D'}_{\alpha_1+\alpha_2})\\
Q &= \exp\left(\frac{4\pi i}{3}\frac{1}{2i}\left(E^D_{\alpha_1}+E^D_{\alpha_2}+E^D_{-\alpha_1-\alpha_2}-E^D_{-\alpha_1}-E^D_{-\alpha_2}-E^D_{\alpha_1+\alpha_2}\right)\right)\,.
\end{aligned} \tag{4.23}$$

Again the subgroup D'' gives rise to an element P'', and the exchange $P' \leftrightarrow P''$ produces a non-equivalent flat connection with the unbroken discrete subgroup.

These E_6-flat connections are essentially the same as the ones for F_4, as can be understood from the decomposition.

$$\begin{aligned}
&E_6 \to F_4\\
&\mathbf{27} \to \mathbf{26}\oplus\mathbf{1}\\
&\mathbf{78} \to \mathbf{52}\oplus\mathbf{26}\,.
\end{aligned} \tag{4.24}$$

Like the embedding of G_2 in SO(7) of [5], it is not hard to make this explicit. Take the roots of E_6 (in the conventions of [5]), set the first and last component of each vector to zero, and renormalise the vectors by multiplying with $2/\sqrt{3}$. This projection gives the roots of F_4. Applying the same projection to the roots (4.20), one finds that the $\mathrm{SU}(3)^3$ subgroup of E_6 projects onto an $\mathrm{SU}(3)^2$ subgroup of F_4.

We furthermore note that, concerning the remark in the previous paragraph on the triples (4.19), that in the fundamental **27** irrep of E_6, the holonomies can be constructed from these triples with the triple $(1,1,1)$ *included.*

4.3.3 E_7

Because the E_6-lattice is a sub-lattice of the E_7-lattice, it is not hard to embed the previous E_6 result into E_7. Working out all details, one finds two inequivalent non-trivial flat connections with an unbroken SU(2), precisely as one would naively expect from the decomposition of E_7 in $F_4 \times \mathrm{SU}(2)$

$$\begin{aligned} E_7 &\to F_4 \times \mathrm{SU}(2) \\ \mathbf{56} &\to (\mathbf{26},\mathbf{2}) \oplus (\mathbf{1},\mathbf{4}) \\ \mathbf{133} &\to (\mathbf{52},\mathbf{1}) \oplus (\mathbf{26},\mathbf{3}) \oplus (\mathbf{1},\mathbf{3})\,. \end{aligned} \tag{4.25}$$

4.3.4 E_8

As the E_6-lattice is also a sublattice of the lattice of E_8, our construction is easily extended to E_8. One finds two inequivalent non-trivial flat connections with unbroken subgroup G_2, in accordance to the decomposition

$$\begin{aligned} E_8 &\to G_2 \times F_4 \\ \mathbf{248} &\to (\mathbf{14},\mathbf{1}) \oplus (\mathbf{7},\mathbf{26}) \oplus (\mathbf{1},\mathbf{52})\,. \end{aligned} \tag{4.26}$$

5. Non-trivial vacua based on twist in SU(4)

It is also possible to use SU(4) for a construction of non-trivial flat connections. However, we need a subtle modification with respect to the SU(2) and SU(3) constructions. After developing the relevant tools for SU(4), we will construct the explicit realisation of this construction in the groups E_7 and E_8.

5.1 Twist in SU(4)

We will take the canonical form (in the conventions of the appendices of [5]), where again we will use H_α and E_α for the generators of the subgroup, and h_α and e_α for the generators of the original group.

In SU(4) we look for two matrices satisfying

$$pq = \exp\left(\frac{2\pi i}{4}\right) qp\,. \tag{5.1}$$

We take:

$$p = \begin{pmatrix} \exp(\frac{3\pi i}{4}) & 0 & 0 & 0 \\ 0 & \exp(\frac{\pi i}{4}) & 0 & 0 \\ 0 & 0 & \exp(-\frac{\pi i}{4}) & 0 \\ 0 & 0 & 0 & \exp(-\frac{3\pi i}{4}) \end{pmatrix}, \qquad q = \begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \end{pmatrix} \exp\left(\frac{\pi i}{4}\right). \tag{5.2}$$

In terms of generators this is

$$p = \exp\left(2\pi i H_{3\alpha_1+4\alpha_2+3\alpha_3}\right) \tag{5.3}$$

$$q = \exp\Big(\frac{\pi i}{2}((1-i)(E_{\alpha_1}+E_{\alpha_2}+E_{\alpha_3}+E_{-\alpha_1-\alpha_2-\alpha_3}) - (E_{\alpha_1+\alpha_2}+E_{\alpha_2+\alpha_3}) - \\ -(E_{-\alpha_1-\alpha_2}+E_{-\alpha_2-\alpha_3}) + (1+i)(E_{-\alpha_1}+E_{-\alpha_2}+E_{-\alpha_3}+E_{\alpha_1+\alpha_2+\alpha_3}))\Big). \tag{5.4}$$

The commutation relations of p and q with the group generators are most easily calculated in a specific representation. One finds

$$\begin{aligned} pH_\alpha &= H_\alpha p\,, & qH_\alpha &= H_{R\alpha} q\,, \\ pE_\alpha &= \exp(2\pi i\langle\alpha, 3\alpha_1+4\alpha_2+3\alpha_4\rangle)E_\alpha p\,, & qE_\alpha &= E_{R\alpha} q\,. \end{aligned} \tag{5.5}$$

The action of the rotation R (which is in this case a combination of a genuine rotation and a reflection) is fully determined by its action on the simple roots.

$$R : \alpha_1 \to \alpha_2 \to \alpha_3 \to -(\alpha_1+\alpha_2+\alpha_3) \to \alpha_1\,. \tag{5.6}$$

Just like in the SU(3)-case this is nicely visualised by a rotation of the extended Dynkin diagram of SU(4). R is an element of the Weyl group: It is the composition of the Weyl reflection generated by α_3, followed by the reflection generated by α_2 and the reflection generated by α_1. It is obvious that $R^4 = 1$, but on the vectors that are a multiple of $\alpha_1+\alpha_3$ R acts as a reflection, and we even have $R^2(\alpha_1+\alpha_3) = \alpha_1+\alpha_3$.

5.2 Calculation of the unbroken subgroup

For the computation of the combinations pT^ap^{-1} and qT^aq^{-1}, we will need several representations of SU(4). For the construction in the fundamental **56** irrep of E_7 we need the **1**, **15** and **6** of SU(4). For the construction in the adjoints of E_7 and E_8 (the **133** resp. **248**) we furthermore need the **10** and **20** of SU(4). Note that the **6** and **10** still contain a non-trivial $\mathbb{Z}_2$ centre, which is a subgroup of the $\mathbb{Z}_4$ centre of their simply connected covering SU(4). To compensate for the non-trivial $\mathbb{Z}_2$, we will use the SU(2) factor in the decomposition $E_7 \to \mathrm{SU}(4)\times\mathrm{SU}(4)\times\mathrm{SU}(2)$. We will need only the one-, two- and three dimensional representation of SU(2), and relevant facts about these representations can be found in our previous article [5].

The calculation of pT^ap^{-1} proceeds in the same way as in (4.7). For qT^aq^{-1} we start again by decomposing the representation of the group G into irreducible representations of SU(4). Construct the normalised eigenvectors ψ_λ of the CSA: $H_\alpha\psi_\lambda = \langle\alpha,\lambda\rangle\psi_\lambda$. It then follows that

$$H_{R\alpha}q\psi_\lambda = qH_\alpha\psi_\lambda = \langle\alpha,\lambda\rangle q\psi_\lambda = \langle R\alpha, R\lambda\rangle q\psi_\lambda\,. \tag{5.7}$$

This means that $q\psi_\lambda$ is an eigenvector of the CSA with weight $R\lambda$. For simple weights, we have $q\psi_\lambda = \phi_\lambda\psi_{R\lambda}$ with ϕ_λ a phase. Because all representations we consider are

JHEP05(1999)014

representations of $\mathrm{SU}(4)/\mathbb{Z}_2$, we have $q^4 = 1$ and we find that $\phi_\lambda \phi_{R\lambda} \phi_{R^2\lambda} \phi_{R^3\lambda} = 1$. With the ladder operators one shows that ϕ_λ is actually independent of λ (in a suitable phase convention). For $\lambda \neq 0$ the following four combinations are eigenvectors:

$$\begin{aligned}
\psi_\lambda + \psi_{R\lambda} + \psi_{R^2\lambda} + \psi_{R^3\lambda} &\to \text{eigenvalue } \phi \\
\psi_\lambda - i\psi_{R\lambda} - \psi_{R^2\lambda} + i\psi_{R^3\lambda} &\to \text{eigenvalue } i\phi \\
\psi_\lambda - \psi_{R\lambda} + \psi_{R^2\lambda} - \psi_{R^3\lambda} &\to \text{eigenvalue } -\phi \\
\psi_\lambda + i\psi_{R\lambda} - \psi_{R^2\lambda} - i\psi_{R^3\lambda} &\to \text{eigenvalue } -i\phi
\end{aligned}$$

Note that for weights λ with $R^2\lambda = \lambda$, two of these combinations are zero.

The **6** and **10** do not posses zero weights, and by checking the eigenvalues we find that the phase $\phi = \pm 1$. The above considerations thus determine the action of q up to an overall sign. In the actual constructions for E_7 and E_8, this sign will be compensated for by another sign coming from an SU(2)-subgroup.

The **15** is the adjoint of SU(4). Among the fifteen weights there are 12 non-zero weights, that decompose in 3 4-cycles under the action of q. The remaining three zero weights correspond to the generators of the CSA, the H_α, on which we already know the action of q. The eigenvectors of q are

$$\begin{aligned}
H_\alpha + H_{R\alpha} + H_{R^2\alpha} + H_{R^3\alpha} &= 0 \\
H_\alpha - iH_{R\alpha} - H_{R^2\alpha} + iH_{R^3\alpha} &\to \text{eigenvalue } i \\
H_\alpha - H_{R\alpha} + H_{R^2\alpha} - H_{R^3\alpha} &\to \text{eigenvalue } -1 \\
H_\alpha + iH_{R\alpha} - H_{R^2\alpha} - iH_{R^3\alpha} &\to \text{eigenvalue } -i\,.
\end{aligned}$$

The last irrep we wish to consider is the **20**. The easiest way to reconstruct the action of q in this irrep, is to use the decomposition of the tensor product of two **6**'s: $\mathbf{6} \otimes \mathbf{6} = \mathbf{1} \oplus \mathbf{15} \oplus \mathbf{20}$. From this decomposition we quickly find that the **20** has 18 non-zero-weights, which under the action of q decompose in 4 4-cycles and one 2-cycle. The two zero weights form a 2-cycle.

We note that these considerations imply that

$$pT^a p^{-1} = \exp\left(\frac{2\pi i n}{4}\right) T^a, \qquad qT^a q^{-1} = \exp\left(\frac{2\pi i m}{4}\right) T^a\,.$$

To compute the representations of SU(4) we follow the same route as for SU(3). Define the CSA generators $H_{\alpha'_1}$, $H_{\alpha'_2}$ and $H_{\alpha'_3}$ associated to the simple roots of su(4), following the conventions of the appendix of [5]. Set $H_{\alpha'_i} = h_{\beta_i}$, for $h_{\beta_i} \in \mathcal{L}_G$. We need the combinations $2\Lambda'_i/\langle \alpha'_k, \alpha'_k \rangle$ (see eq. (3.5)), where Λ'_i are the fundamental weights of su(4) and α'_k are simple roots of su(4). In our conventions $\langle \alpha'_k, \alpha'_k \rangle = 1/4$, for α'_k a root of su(4). The combinations we need are thus

$$8\Lambda'_1 = 6\beta_1 + 4\beta_2 + 2\beta_3\,, \qquad 8\Lambda'_2 = 4\beta_1 + 8\beta_2 + 4\beta_3\,, \qquad 8\Lambda'_3 = 2\beta_1 + 4\beta_2 + 6\beta_3\,.$$

The q_k from eq. (3.5) can then be found by computing the inner products of $8\Lambda'_i$ with the weights of G.

JHEP05(1999)014

5.3 Realisations of the SU(4)-based construction

5.3.1 E_7

As mentioned before, we need the subgroup of E_7 whose universal cover is $\mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{SU}(4)$. This subgroup can be chosen to be regular, such that its lattice is a sublattice of the E_7-lattice. We take as SU(2)-root:

$$\frac{(\sqrt{2},0,0,0,0,0,0)}{6\,.} \tag{5.8}$$

For the first SU(4) factor that we will label $\mathrm{SU}(4)_1$, we take as roots:

$$\alpha_1^1 = \frac{(0,0,0,0,0,1,1)}{6}\,, \qquad \alpha_2^1 = \frac{(0,0,0,0,1,-1,0)}{6}\,, \qquad \alpha_3^1 = \frac{(0,0,0,0,0,1,-1)}{6}\,. \tag{5.9}$$

The second SU(4)-factor ($\mathrm{SU}(4)_2$) has as roots:

$$\alpha_1^2 = \frac{(0,0,1,1,0,0,0)}{6}\,, \qquad \alpha_2^2 = \frac{(0,1,-1,0,0,0,0)}{6}\,, \qquad \alpha_3^2 = \frac{(0,0,1,-1,0,0,0)}{6}\,. \tag{5.10}$$

With these roots we have the following branching rules for E_7 into $\mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{SU}(4)$

$$\begin{aligned}
E_7 &\to \mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{SU}(4)\\
\mathbf{56} &\to (\mathbf{1},\mathbf{4},\bar{\mathbf{4}}) \oplus (\mathbf{1},\bar{\mathbf{4}},\mathbf{4}) \oplus (\mathbf{2},\mathbf{6},\mathbf{1}) \oplus (\mathbf{2},\mathbf{1},\mathbf{6})\\
\mathbf{133} &\to (\mathbf{1},\mathbf{15},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{15}) \oplus (\mathbf{3},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{6},\mathbf{6}) \oplus (\mathbf{2},\mathbf{4},\mathbf{4}) \oplus (\mathbf{2},\bar{\mathbf{4}},\bar{\mathbf{4}})\,.
\end{aligned} \tag{5.11}$$

In the $\mathrm{SU}(4) \times \mathrm{SU}(4)$-part we construct as usual a diagonal subgroup D. We only list the Cartan generators:

$$\begin{aligned}
H^D_{\alpha_1} &= \frac{3}{4} h_{(0,0,1,1,0,1,1)}\\
H^D_{\alpha_2} &= \frac{3}{4} h_{(0,1,-1,0,1,-1,0)}\\
H^D_{\alpha_3} &= \frac{3}{4} h_{(0,0,1,-1,0,1,-1)}\,.
\end{aligned} \tag{5.12}$$

The SU(2)-factor has as its generators

$$L_3 = 3h_{(\sqrt{2},0,0,0,0,0,0)}\,, \qquad L_+ = \sqrt{18} e_{(\sqrt{2},0,0,0,0,0,0)}\,. \tag{5.13}$$

We compute the inner products of the weights of E_7 with $(3\sqrt{2},0,0,0,0,0,0)$ to find the SU(2)-representations (if the inner product is integer, the corresponding weight belongs to an SO(3) representation, otherwise it is a genuine SU(2) representation with $\mathbb{Z}_2$-centre [5]). To check the SU(4) irreps, it is sufficient to compute q_1 (see (3.5)) for the diagonal group. q_1 is found by computing the inner product of the weights of

E_7 with $6\alpha_1^D+4\alpha_2^D+2\alpha_3^D = (0,3,3,3,3,3,3)$ (where $\alpha_i^D = \alpha_i^1+\alpha_i^2$ are the simple roots for the diagonal group). If $q_1 \bmod 1 = 3/4$ the weight belongs to a representation of the fundamental SU(4), for $q_1 \bmod 1 = 1/4$ it belongs to an irrep that is congruent to the complex conjugate of the fundamental, for $q_1 \bmod 1 = 1/2$ the irrep is congruent to $\mathrm{SU}(4)/\mathbb{Z}_2$, and if q_1 is an integer the irrep is congruent to $\mathrm{SU}(4)/\mathbb{Z}_4$. Careful examination shows that all weights of E_7 fall into two categories: One category is formed by weights that are in an irrep of $(\mathrm{SU}(4)/\mathbb{Z}_4) \times \mathrm{SO}(3)$, while the second category consist of weights that are in an faithful irrep of $(\mathrm{SU}(4)/\mathbb{Z}_2) \times \mathrm{SU}(2)$. We now construct the elements

$$
\begin{aligned}
P_{\mathrm{SU}(4)} &= \exp\left(2\pi i H^D_{3\alpha_1+4\alpha_2+3\alpha_3}\right)\\
Q_{\mathrm{SU}(4)} &= \exp\frac{\pi i}{2}\Big((1-i)\left(E^D_{\alpha_1}+E^D_{\alpha_2}+E^D_{\alpha_3}+E^D_{-\alpha_1-\alpha_2-\alpha_3}\right)-\left(E^D_{\alpha_1+\alpha_2}+E^D_{\alpha_2+\alpha_3}\right)-\\
&\qquad -\left(E^D_{-\alpha_1-\alpha_2}+E^D_{-\alpha_2-\alpha_3}\right)+(1+i)\left(E^D_{-\alpha_1}+E^D_{-\alpha_2}+E^D_{-\alpha_3}+E^D_{\alpha_1+\alpha_2+\alpha_3}\right)\Big)
\end{aligned}
$$

in the diagonal SU(4)-group, and

$$
P_{\mathrm{SU}(2)} = \exp(i\pi L_3)\,, \qquad Q_{\mathrm{SU}(2)} = \exp\left(\frac{i\pi}{2}(L_+ + L_-)\right) \tag{5.14}
$$

in the SU(2) group. For a weight λ with corresponding eigenvector ψ_λ, that belongs to an $(\mathrm{SU}(4)/\mathbb{Z}_4)\times\mathrm{SO}(3)$-irrep, we have

$$
P_{\mathrm{SU}(4)}Q_{\mathrm{SU}(4)}\psi_\lambda = Q_{\mathrm{SU}(4)}P_{\mathrm{SU}(4)}\psi_\lambda\,, \qquad P_{\mathrm{SU}(2)}Q_{\mathrm{SU}(2)}\psi_\lambda = Q_{\mathrm{SU}(2)}P_{\mathrm{SU}(2)}\psi_\lambda\,.
$$

If the weight λ belongs to a faithful $(\mathrm{SU}(4)/\mathbb{Z}_2)\times\mathrm{SU}(2)$-irrep, we have

$$
P_{\mathrm{SU}(4)}Q_{\mathrm{SU}(4)}\psi_\lambda = -Q_{\mathrm{SU}(4)}P_{\mathrm{SU}(4)}\psi_\lambda\,, \qquad P_{\mathrm{SU}(2)}Q_{\mathrm{SU}(2)}\psi_\lambda = -Q_{\mathrm{SU}(2)}P_{\mathrm{SU}(2)}\psi_\lambda\,.
$$

The products

$$
P = P_{\mathrm{SU}(2)}P_{\mathrm{SU}(4)} = \exp(3\pi i h_{(\sqrt{2},2,1,0,2,1,0)})\,,, \qquad Q = Q_{\mathrm{SU}(2)}Q_{\mathrm{SU}(4)} \tag{5.15}
$$

are commuting elements of E_7, since they commute on each weight of any representation. We construct another diagonal subgroup D' in $\mathrm{SU}(4)\times\mathrm{SU}(4)$ as usual, by applying the rotation

$$
R: \alpha_1 \to \alpha_2 \to \alpha_3 \to -(\alpha_1+\alpha_2+\alpha_3) \to \alpha_1\,, \tag{5.16}
$$

to the first SU(4) factor and then reconstructing the diagonal subgroup. In this diagonal subgroup we construct an element

$$
P'_{\mathrm{SU}(4)} = \exp\left(i\pi H^{D'}_{3\alpha_1+4\alpha_2+3\alpha_3}\right)\,. \tag{5.17}
$$

The same considerations as above lead to the conclusion that the element

$$P' = P_{\mathrm{SU}(2)} P'_{\mathrm{SU}(4)} = \exp\left(3\pi i h_{(\sqrt{2},0,-1,-2,2,1,0)}\right) \tag{5.18}$$

commutes with P and Q, and hence we have three candidate holonomies $\Omega_1 = P$, $\Omega_2 = P'$ and $\Omega_3 = Q$.

Next we calculate the unbroken subgroup. Computing the commutators of the generators of E_7 with P and P', we find that only the CSA-generators of E_7 commute with both P and P'. These CSA generators correspond to zero weights in the representation of the $\mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{SU}(4)$ subgroup, and from previous considerations we know that there is no combination of zero weights of the adjoints of SU(4) or SU(2) that commutes with Q. Hence we find that there is no generator that commutes with P, P' and Q, and the unbroken subgroup is at most discrete.

Instead of the rotation R (5.16), one could also apply the rotations R^2 or R^3 to construct elements that we will call P'' resp. P'''. The holonomies $\Omega_1 = P$, $\Omega_2 = P'''$ and $\Omega_3 = Q$ also only commute with a discrete subgroup of E_7, and give a non-trivial flat connection inequivalent to the one implied by $\Omega_1 = P$, $\Omega_2 = P'$ and $\Omega_3 = Q$. The non-trivial flat connection implied by $\Omega_1 = P$, $\Omega_2 = P''$ and $\Omega_3 = Q$ gives a bigger unbroken subgroup. The unbroken symmetry group can be calculated to be $U(1)^3$, and this non-trivial flat connection is actually a gauge deformation of the flat connection constructed in [5], based on twist in SU(2). Therefore the construction based on twist in $\mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{SU}(4)$ gives two new vacua with discrete symmetry group, and a gauge deformation of a vacuum configuration that we have encountered before.

We wish to point out that also the SU(4)-based construction in E_7 fits into a pattern. We already remarked that the G_2 and SO(7) non-trivial flat connection can be described by the 7 triples $(\pm 1, \pm 1, \pm 1)$ with at least one -1, and that the SU(3)-based construction in F_4 can be characterised by the 26 triples (4.19) with at least one element not equal to 1. For the SU(4)-based construction, the eigenvalues that appear on the diagonal of the diagonalised holonomies are i^n. Consider now the triples $(i^{n_1}, i^{n_2}, i^{n_3})$ with $n_j \in \mathbb{Z}$. There are $4^3 = 64$ distinct triples. Now exclude all triples that are not of order 4, by which we mean that we demand each triple to contain at least one i or $-i$. We are then left with $64 - 2^3 = 56$ triples. 56 is precisely the dimension of the fundamental irrep of E_7, and indeed, constructing the holonomies in this representation and diagonalising, we find that the triples $((\Omega_1)_{ii}, (\Omega_2)_{ii}, (\Omega_3)_{ii})$ are precisely the triples mentioned above.

5.3.2 E_8

The $\mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{SU}(4)$ construction of E_7 can be easily embedded in E_8 by using the fact that the E_7-lattice is a sublattice of the E_8-lattice. More specific, we can take as SU(2)-root:

$$\frac{(1,1,0,0,0,0,0,0)}{\sqrt{60}} \tag{5.19}$$

One SU(4) factor has as roots:

$$\alpha_1 = \frac{(0,0,0,0,0,0,1,1)}{\sqrt{60}}$$
$$\alpha_2 = \frac{(0,0,0,0,0,1,-1,0)}{\sqrt{60}}$$
$$\alpha_3 = \frac{(0,0,0,0,0,0,1,-1)}{\sqrt{60}} \tag{5.20}$$

A second SU(4)-factor is generated by the roots:

$$\alpha_1 = \frac{(0,0,0,1,1,0,0,0)}{\sqrt{60}}$$
$$\alpha_2 = \frac{(0,0,1,-1,0,0,0,0)}{\sqrt{60}}$$
$$\alpha_3 = \frac{(0,0,0,1,-1,0,0,0)}{\sqrt{60}} \tag{5.21}$$

With this information it is trivial to copy the E_7 construction. Like in the E_7-case, one finds 3 vacua. For two of these, the subgroup unbroken by the E_7 holonomies is SU(2). This is to be expected, since E_8 decomposes into $E_7 \times \mathrm{SU}(2)$:

$$\begin{aligned} E_8 &\to E_7 \times \mathrm{SU}(2) \\ \mathbf{248} &\to (\mathbf{133},\mathbf{1}) \oplus (\mathbf{56},\mathbf{2}) \oplus (\mathbf{1},\mathbf{3}) \end{aligned} \tag{5.22}$$

The third vacuum is, as for E_7, a gauge deformation of the vacuum based on twist in SU(2) constructed in [5].

6. Non-trivial vacua based on twist in SU(5)

6.1 Twist in SU(5)

We will take the canonical form (in the conventions of the appendices of [5]), where again we will use H_α and E_α for the subgroup, and h_α and e_α) for the original group.

In SU(5) we look for two matrices satisfying

$$pq = \exp\left(\frac{2\pi i}{5}\right) qp\,. \tag{6.1}$$

We take:

$$p = \begin{pmatrix} \exp(\frac{4\pi i}{5}) & 0 & 0 & 0 & 0 \\ 0 & \exp(\frac{2\pi i}{5}) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \exp(-\frac{2\pi i}{5}) & 0 \\ 0 & 0 & 0 & 0 & \exp(-\frac{4\pi i}{5}) \end{pmatrix}, \qquad q = \begin{pmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 1&0&0&0&0 \end{pmatrix}. \tag{6.2}$$

In terms of generators this is

$$p = \exp\left(2\pi i H_{4\alpha_1+6\alpha_2+6\alpha_3+4\alpha_4}\right) \tag{6.3}$$

$$\begin{aligned} q = \exp\Big(\frac{2\pi i}{5}\big(&a(E_{\alpha_1}+E_{\alpha_2}+E_{\alpha_3}+E_{\alpha_4}+E_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)})+\\ &+b(E_{\alpha_1+\alpha_2}+E_{\alpha_2+\alpha_3}+E_{\alpha_3+\alpha_4}+E_{-(\alpha_1+\alpha_2+\alpha_3)}+E_{-(\alpha_2+\alpha_3+\alpha_4)})+\\ &+\text{complex conjugate}\big)\Big)\,, \end{aligned} \tag{6.4}$$

with

$$a = \left[\exp\left(-\frac{2\pi i}{5}\right)-\exp\left(\frac{2\pi i}{5}\right)\right]+2\left[\exp\left(-\frac{4\pi i}{5}\right)-\exp\left(\frac{4\pi i}{5}\right)\right] \tag{6.5}$$

$$b = \left[\exp\left(-\frac{4\pi i}{5}\right)-\exp\left(\frac{4\pi i}{5}\right)\right]+2\left[\exp\left(\frac{2\pi i}{5}\right)-\exp\left(-\frac{2\pi i}{5}\right)\right]. \tag{6.6}$$

The commutation relations of p and q with the group generators are most easily calculated in a specific representation. One finds

$$\begin{aligned} pH_\alpha &= H_\alpha p\,, & qH_\alpha &= H_{R\alpha}q\,,\\ pE_\alpha &= \exp(2\pi i\langle\alpha, 4\alpha_1+6\alpha_2+6\alpha_3+4\alpha_4\rangle)E_\alpha p\,, & qE_\alpha &= E_{R\alpha}q\,. \end{aligned} \tag{6.7}$$

The action of the rotation R (which is in this case a genuine rotation) is fully determined by its action on the simple roots.

$$R:\alpha_1\to\alpha_2\to\alpha_3\to\alpha_4\to-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\to\alpha_1\,. \tag{6.8}$$

Again this can be conveniently depicted by a permutation of the roots of the extended Dynkin diagram of SU(5). R is an element of the Weyl group: It is the composition of the Weyl reflection generated by α_4, followed by the reflections generated by α_3, α_2, and α_1.

For the calculation of the unbroken subgroup we will only need the action of q on the zero weights of the adjoint. These are the CSA-generators, on which the action of q is easily diagonalised. The only fact we will need is that there is no linear combination of CSA-generators that commutes with q.

6.2 Realisation of the SU(5)-based construction

6.2.1 E_8

Only E_8 allows a suitable PG'_2-subgroup with $\widetilde{PG'}_2 = \mathrm{SU}(5)\times\mathrm{SU}(5)$. We take as root vectors for the first SU(5)-subgroup

$$\begin{aligned} \alpha^1_1 &= \left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)\frac{1}{\sqrt{60}}\,, & \alpha^1_2 &= \frac{(0,0,0,0,0,0,1,1)}{\sqrt{60}}\,,\\ \alpha^1_3 &= \frac{(0,0,0,0,0,1,-1,0)}{\sqrt{60}}\,, & \alpha^1_4 &= \frac{(0,0,0,0,0,0,1,-1)}{\sqrt{60}}\,, \end{aligned} \tag{6.9}$$

while the second SU(5)-factor will have roots

$$
\begin{aligned}
\alpha_1^2 &= \frac{(0,0,0,1,-1,0,0,0)}{\sqrt{60}}\,, \qquad & \alpha_2^2 &= \frac{(0,0,1,-1,0,0,0,0)}{\sqrt{60}}\,, \\
\alpha_3^2 &= \frac{(0,1,-1,0,0,0,0,0)}{\sqrt{60}}\,, \qquad & \alpha_4^2 &= \frac{(-1,-1,0,0,0,0,0,0)}{\sqrt{60}}\,.
\end{aligned} \tag{6.10}
$$

The decomposition of E_8 into this $\mathrm{SU}(5)\times\mathrm{SU}(5)$ for the adjoint is given by[1]

$$
\begin{aligned}
E_8 &\to \mathrm{SU}(5)\times\mathrm{SU}(5) \\
\mathbf{248} &\to (\mathbf{24},\mathbf{1})\oplus(\mathbf{1},\mathbf{24})\oplus(\mathbf{10},\mathbf{5})\oplus(\mathbf{5},\overline{\mathbf{10}})\oplus(\bar{\mathbf{5}},\mathbf{10})\oplus(\overline{\mathbf{10}},\bar{\mathbf{5}})\,.
\end{aligned} \tag{6.11}
$$

The congruence classes of the irreducible components of $\mathrm{SU}(5)\times\mathrm{SU}(5)$ are then[1] $(0,0)$ with multiplicity 2, and $(3,1)$, $(1,2)$, $(4,3)$ and $(2,4)$ all with multiplicity 1. We note that for each congruence class (a,b) we have $a+2b \bmod 5 = 0$. Since all E_8 representations are isomorphic, this must automatically hold for any E_8 representation.

We now construct the elements

$$
P_i = \exp\left(2\pi i H_{4\alpha_1^i+6\alpha_2^i+6\alpha_3^i+4\alpha_4^i}\right) \tag{6.12}
$$

$$
\begin{aligned}
Q_i = \exp\Bigg(\frac{2\pi i}{5}\Big(&a(E_{\alpha_1^i}+E_{\alpha_2^i}+E_{\alpha_3^i}+E_{\alpha_4^i}+E_{-(\alpha_1^i+\alpha_2^i+\alpha_3^i+\alpha_4^i)})+ \\
&+b(E_{\alpha_1^i+\alpha_2^i}+E_{\alpha_2^i+\alpha_3^i}+E_{\alpha_3^i+\alpha_4^i}+E_{-(\alpha_1^i+\alpha_2^i+\alpha_3^i)}+E_{-(\alpha_2^i+\alpha_3^i+\alpha_4^i)})+ \\
&+\text{complex conjugate}\Big)\Bigg)\,.
\end{aligned} \tag{6.13}
$$

When acting on an eigenvector ψ_λ, corresponding to a weight λ that belongs to an irrep with $\mathrm{SU}(5)\times\mathrm{SU}(5)$-congruence class labelled by (n_1,n_2), P_i and Q_i obey the following commutation rule

$$
P_iQ_i\psi_\lambda = \exp\left(\frac{2\pi i n_i}{5}\right)Q_iP_i\psi_\lambda\,. \tag{6.14}
$$

From the above observation on congruence classes it follows that for any weight λ of E_8 we have

$$
P_1Q_1(P_2)^2Q_2\psi_\lambda = \exp\left(\frac{2\pi i(n_1+2n_2)}{5}\right)Q_1P_1Q_2(P_2)^2\psi_\lambda = Q_1P_1Q_2(P_2)^2\psi_\lambda\,. \tag{6.15}
$$

Hence the elements

$$
P = P_1(P_2)^2\,, \qquad Q = Q_1Q_2 \tag{6.16}
$$

[1]We will label the irrep with Dynkin labels (1000) as **5** and the irrep with labels (0010) as **10**. We define the congruence class of the representation with Dynkin labels (n_1,n_2,n_3,n_4) to be labelled by the integer $n_1+2n_2+3n_3+4n_4 \bmod 5$.

JHEP05(1999)014

commute. A third commuting element is constructed in the standard way:

$$P' = Q_1^{-n} P Q_1^n = Q_2^n P Q_2^{-n} \tag{6.17}$$

n can range from 1 to 4, and therefore there are 4 different flat connections constructed this way. We will work out the case where $n = 1$.

In that case we have

$$P = \exp\left(\frac{2\pi i}{5}\sqrt{15} h_{(-6,2,-2,-6,-10,4,2,0)}\right) \tag{6.18}$$

$$P' = \exp\left(\frac{2\pi i}{5}\sqrt{15} h_{(-2,-2,-6,-10,6,4,2,0)}\right) \tag{6.19}$$

After a somewhat tedious calculation we find that none of the ladder operators e_β of E_8 commutes with both P and P', so we are only left with the 8 Cartan generators. We have to check whether these commute with Q. Q has been constructed in such a way that it is an element of the diagonal SU(5)-subgroup, so we can apply our previous methods. The elements of the Cartan subalgebra are also the CSA elements for SU(5) $\times$ SU(5), and we know that none of these are invariant under commutation with Q (Q takes the role of q in the diagonal subgroup of SU(5) $\times$ SU(5)). Hence no group generator commutes with P, P' and Q simultaneously, and the unbroken subgroup is at most discrete.

For the SU(5) based construction, the eigenvalues that appear in the diagonalised holonomies are $\exp\left(\frac{2\pi i n}{5}\right)$. Consider now the triples

$$(\exp(\frac{2\pi i n_1}{5}), \exp(\frac{2\pi i n_2}{5}), \exp(\frac{2\pi i n_3}{5})) \qquad n_i \in \mathbb{Z} \tag{6.20}$$

If one excludes the triple $(1, 1, 1)$, there are $5^3 - 1 = 124$ distinct triples of this type. Unlike the cases we considered previously, 124 is not the dimension of any representation of E_8. The smallest non-trivial representation of E_8 is the 248-dimensional adjoint. Constructing the holonomies in the adjoint and diagonalising, one finds that the 248 triples $(P_{ii}, P'_{ii}, Q_{ii})$ consist of precisely two sets of the above 124 triples.

7. Non-trivial vacua based on twist in SU(2) $\times$ SU(3)

At first sight all possibilities seem to be exhausted, since constructions with SU(N), $N > 5$ are impossible because the exceptional groups are not big enough to contain more than one SU(6)-factor. There is however still one more possibility, based on our previous constructions based on SU(2) and SU(3). It is easy to see that E_8 allows the SU(2) non-trivial flat connection to be realised simultaneously with the SU(3)-flat connection.

7.1 E_8

According to the decomposition (4.26) E_8 allows a $G_2 \times F_4$ subgroup (in this case we are dealing with a subgroup that is a genuine product). In [5] we constructed holonomies for a non-trivial flat connection in G_2, that we named P, P' and Q. Here we will rename them to P_{G_2}, P'_{G_2} and Q_{G_2} to avoid confusion. F_4 allows different types of non-trivial flat connections. We will use the non-trivial flat connections constructed in section 4.3.1, and will rename the holonomies P, P', P'' and Q constructed there to P_{F_4}, P'_{F_4}, P''_{F_4} and Q_{F_4}. Decomposing E_8 into its $G_2 \times F_4$ subgroup, elements of this subgroup can be denoted by a pair of elements (g_{G_2}, g_{F_4}) with $g_{G_2} \in G_2$ and $g_{F_4} \in F_4$. Therefore (P_{G_2}, P_{F_4}) and (Q_{G_2}, Q_{F_4}) represent commuting elements of E_8. A third commuting element can be constructed by applying the standard procedure of rotating group factors to both elements of the pair to construct the element (P'_{G_2}, P'_{F_4}). The rotation can be applied multiple times to construct multiple triples. We have the following possibilities:

$$\Omega_1 = (P_{G_2}, P_{F_4})\,, \qquad \Omega_2 = (P'_{G_2}, P'_{F_4})\,, \qquad \Omega_3 = (Q_{G_2}, Q_{F_4})\,, \tag{7.1}$$

$$\Omega_1 = (P_{G_2}, P_{F_4})\,, \qquad \Omega_2 = (P_{G_2}, P''_{F_4})\,, \qquad \Omega_3 = (Q_{G_2}, Q_{F_4})\,, \tag{7.2}$$

$$\Omega_1 = (P_{G_2}, P_{F_4})\,, \qquad \Omega_2 = (P'_{G_2}, P_{F_4})\,, \qquad \Omega_3 = (Q_{G_2}, Q_{F_4})\,, \tag{7.3}$$

$$\Omega_1 = (P_{G_2}, P_{F_4})\,, \qquad \Omega_2 = (P_{G_2}, P'_{F_4})\,, \qquad \Omega_3 = (Q_{G_2}, Q_{F_4})\,, \tag{7.4}$$

$$\Omega_1 = (P_{G_2}, P_{F_4})\,, \qquad \Omega_2 = (P'_{G_2}, P''_{F_4})\,, \qquad \Omega_3 = (Q_{G_2}, Q_{F_4})\,. \tag{7.5}$$

However, not all of these are new. The flat connection implied by the holonomies in (7.2) has twice the element P_{G_2} in the G_2-factor: This means that in the G_2 subgroup the connection can be deformed to a trivial one, and only the F_4 part is non-trivial. This is a deformation of one of the flat connections based on SU(3)-twist that was already constructed previously. Similarly the flat connection implied by the holonomies (7.4) is deformable to the other flat connection based on SU(3)-twist. The flat connection implied by (7.3) is trivial in its F_4 factor, and only non-trivial in its G_2 part. It is therefore a deformation of the flat connection in E_8 that was constructed in [5]. The remaining two flat connections are new. We note that, to commute a generator with the Ω_i, we can commute it first with the G_2-element and then with the F_4-element. We can choose a basis of generators such that they commute or anti-commute with the G_2-elements, and they commute up to a $\mathbb{Z}_3$-element with the F_4-elements. We conclude that to commute with the Ω_i a generator has to commute with both the G_2 and the F_4-elements. The generators that commute with the G_2 part of the holonomies generate the F_4 subgroup in the decomposition $G_2 \times F_4$. The F_4 part of the holonomies breaks this F_4 group completely. Hence only discrete symmetries commute with the holonomies constructed here.

JHEP05(1999)014

8. Re-evaluation of Tr$(-)^F$

One of the motivations for this research was to check whether non-trivial flat connections may solve the problem of calculating Tr$(-)^F$ in supersymmetric gauge theories with exceptional gauge groups. We announced in [5] that it does, and we will now justify this claim.

We have shown that the bosonic equations of motion allow several solutions, to be divided in several components of the moduli space. Upon quantisation one finds that each component implies a bosonic vacuum [1, 3]. To perform the vacuum count one has to count in how many ways zero-energy fermions can be added. The reasoning of [1] implies that in each case there are r ways to add fermion pairs to the vacuum, with r the rank of the subgroup that is unbroken on the specific component of moduli space. This gives r bosonic vacua, which together with the original vacuum based on a configuration with bosonic fields only gives a contribution of $r + 1$ to Tr$(-)^F$. We now summarise the results of this paper and [5]:

- F_4: We have found 4 components of the moduli space of F_4:
  - As always there is the component of the trivial flat connections. At a generic point there is a rank 4 unbroken subgroup, giving a contribution of $r + 1 = 5$ to Tr$(-)^F$.
  - In [5] we constructed a new component in moduli space that has maximal unbroken symmetry group SO(3). By deforming around this solution one finds a component of moduli space that has a rank 1 unbroken subgroup, giving a contribution of $r + 1 = 2$ to Tr$(-)^F$.
  - As announced in [5], and explicitly constructed here, there are two components in moduli space that have unbroken discrete symmetry group. These solutions cannot be deformed, and give contributions of $r + 1 = 1$ to Tr$(-)^F$.

  The Witten index count for F_4 SYM on $T^3 \times R$ thus gives

  $$\mathrm{Tr}(-)^F = 5 + 2 + (1 + 1) = 9\,,$$

  in full agreement with the infinite volume calculation since the dual Coxeter number $h = 9$ for F_4

- E_6: We have found 4 components of the moduli space of E_6:
  - The component of the trivial flat connections. At a generic point there is a rank 6 subgroup unbroken, giving a contribution of $r+1 = 7$ to Tr$(-)^F$.

- In [5] we constructed a point in moduli space that has maximal unbroken symmetry group SU(3). By deforming around this solution one finds a component of moduli space that has a rank 2 subgroup unbroken, giving a contribution of $r+1=3$ to $\mathrm{Tr}(-)^F$.
- In this article we explicitly constructed two components in moduli space that have unbroken discrete symmetry group. These solutions cannot be deformed, and each contribute $r+1=1$ to $\mathrm{Tr}(-)^F$.

The Witten index count for E_6 SYM on $T^3 \times R$ thus gives

$$\mathrm{Tr}(-)^F = 7+3+(1+1) = 12\,,$$

in full agreement with the infinite volume calculation since the dual Coxeter number $h=12$ for E_6.

- E_7: We have found 6 components of the moduli space of E_7:
  - The component of the trivial flat connections. At a generic point there is a rank 7 subgroup unbroken, giving a contribution of $r+1=8$ to $\mathrm{Tr}(-)^F$.
  - In [5] we constructed a point in moduli space that has maximal unbroken symmetry group Sp(3). By deforming around this solution one finds a component of moduli space that has a rank 3 subgroup unbroken, giving a contribution of $r+1=4$ to $\mathrm{Tr}(-)^F$.
  - In this article we explicitly constructed, based on twist in SU(3)-subgroups, two components in moduli space with maximal unbroken symmetry group SU(2). Deformations leave a rank 1 unbroken subgroup. We get contributions of each $r+1=2$ to $\mathrm{Tr}(-)^F$.
  - Based on twist in the $\mathrm{SU}(4)\times\mathrm{SU}(4)\times\mathrm{SU}(2)$-subgroups, two more components were constructed that consist of a point in moduli space that has unbroken discrete symmetry group. These solutions cannot be deformed, and each contributes $r+1=1$ to $\mathrm{Tr}(-)^F$.

  The Witten index count for E_7 SYM on $T^3 \times R$ thus gives

  $$\mathrm{Tr}(-)^F = 8+4+(2+2)+(1+1) = 18\,,$$

  in full agreement with the infinite volume calculation since the dual Coxeter number $h=18$ for E_7.

- E_8: We have found 12 components of the moduli space of E_8:
  - The component of the trivial flat connections. At a generic point there is a rank 8 subgroup unbroken, giving a contribution of $r+1=9$ to $\mathrm{Tr}(-)^F$.

– In [5] we constructed a point in moduli space that has unbroken symmetry group F_4. By deforming around this solution one finds a component of moduli space that has a rank 4 subgroup unbroken, giving a contribution of $r + 1 = 5$ to $\mathrm{Tr}(-)^F$.
– In this article, based on twist in SU(3)-subgroups, two components in moduli space were constructed with maximal unbroken symmetry group G_2. At a generic point of each of the two components there is a rank 2 subgroup unbroken. Each component contributes $r + 1 = 3$ to $\mathrm{Tr}(-)^F$.
– Based on twist in the $\mathrm{SU}(4)\times\mathrm{SU}(4)\times\mathrm{SU}(2)$-subgroups, two more components in moduli space were found with maximal unbroken SU(2) symmetry group. Deformations always leave a rank 1 subgroup unbroken. The components contribute each $r + 1 = 2$ to $\mathrm{Tr}(-)^F$.
– With twist in the $\mathrm{SU}(5) \times \mathrm{SU}(5)$-subgroup, four more components were constructed with unbroken discrete symmetry group. These solutions cannot be deformed, and contribute each $r + 1 = 1$ to $\mathrm{Tr}(-)^F$.
– Last, by using $(\mathrm{SU}(2) \times \mathrm{SU}(3))^2$-subgroups, two more components were constructed that have unbroken discrete symmetry group. These solutions can also not be deformed, and contribute each $r + 1 = 1$ to $\mathrm{Tr}(-)^F$.

The Witten index count for E_8 SYM on $T^3 \times R$ thus gives

$$\mathrm{Tr}(-)^F = 9 + 5 + (3 + 3) + (2 + 2) + (1 + 1 + 1 + 1) + (1 + 1) = 30$$

in agreement with the infinite volume calculation since the dual Coxeter number $h = 30$ for E_8.

9. Discussion and conclusions

The construction of non-trivial flat connections with our method reveals that the moduli spaces of exceptional gauge theories on $T^3 \times R$ consists of many components. The Witten index count indicates that all non-trivial flat connections have been found. Thus for supersymmetric Yang-Mills theory both the two extreme cases of the infinite volume (R^4) and small volume ($T^3 \times R$) have h vacua, with h the dual Coxeter number of the gauge group. It would be interesting to learn more about the intermediate volume regime, and the character of the h vacua there. More open questions are the Chern-Simons functional of the gauge field configuration, and the chirality assignment of the different disconnected components.

An intriguing question is what happens for higher dimensional tori. It is clear that the non-trivial flat connections for the 3–torus can always be trivially embedded on an n–torus, by taking $(n - 3)$ holonomies equal to the identity, and the remaining 3 holonomies given by our construction. However, also more general configurations

JHEP05(1999)014

are possible. Our method is basically a construction of commuting elements for certain gauge groups, and a set of n commuting elements may be used as holonomies on an n–torus. For SU(N) and Sp(N) theories the moduli space always consists of one component, regardless of n, but for other groups the number of components of moduli space might grow further with n. Actually, for the orthogonal groups, it is not too hard to see that this is indeed the case.

As an example consider SO(15)-gauge theory on the 4–torus. In the same fashion as for the 3–torus [3], a flat connection for this theory is described by a $\mathbb{Z}_2$-invariant configuration of 15 D-branes on $\tilde{T}^4$, which is the covering space of an orientifold $\tilde{T}^4/\mathbb{Z}_2$. There are 16 orientifold planes, corresponding to all possible combinations of eigenvalues $(\pm1, \pm1, \pm1, \pm1)$ of the four holonomies around the non-trivial cycles. Now distribute the 15 D-branes over the 16 orientifold planes, one brane at each plane, leaving only the orientifold plane that corresponds to the eigenvalues $(1, 1, 1, 1)$ empty. Like in [3], one can compute the total Stieffel-Whitney class of this configuration. The easiest way to do this is to divide the branes into two groups, the first group consisting of the 7 branes at $(\pm1, \pm1, \pm1, 1)$, the second group containing the remaining 8 branes. The first 7 branes are in the same configuration that Witten used to prove the existence of a non-trivial flat connection on the 3–torus (embedded in a higher-dimensional orientifold) and hence these do not contribute to the total Stieffel-Whitney class. The remaining 8 also do not contribute, as can be explicitly calculated. Hence the total Stieffel-Whitney class is trivial, and the bundle is topologically trivial. Using the same arguments as in [3], it is also easily shown that this configuration gives an isolated point in the moduli space of flat SO(15)-connections on T^4, and that the gauge group is completely broken. This generalises to still higher dimensional tori: $2^n - 1$ branes distributed over 2^n orientifold planes in the covering of the orientifold $\tilde{T}^n/\mathbb{Z}_2$, one brane at each plane but leaving the plane corresponding to $(1, 1, \ldots, 1)$ empty, corresponds to a topologically trivial flat SO($2^n - 1$)-connection on T^n (calculation of the total Stieffel-Whitney class of this configuration is trivial, since the $2^n - 1$-branes can be divided in one group of 7, and $2^{n-3} - 1$ groups of 8, where in each group $(n - 3)$ eigenvalues of the holonomies are fixed. The group of 7 does not contribute, nor does any of the groups of 8, giving a trivial total Stieffel-Whitney class). We see that for SO(N) with $N \geq 2^n - 1$, the moduli space of flat SO(N)-connections on T^n has at least one component more than the moduli space of flat SO(N)-connections on T^{n-1}.

We expect that the flat connections constructed in this paper are of relevance in toroidal compactifications of the $E_8 \times E_8$ heterotic string, just like the original constructions of [3] are of relevance for the string theories with SO(32) gauge groups. E_8 has a rich moduli space of 12 components, but our analysis shows that many components survive for exceptional subgroups of E_8. The appearance of disconnected components in the moduli space, and the reduction of the rank of the gauge group show that these theories are not contained in the standard Narain-compactifications of the heterotic string [10]. This is easily understood, since Narain compactification assumes Wilson lines with gauge fields taking values in the CSA, and can therefore only reproduce what we have called trivial flat connections.

Another question is what happens to all these vacuum solutions under various dualities. The existence of disconnected components and rank reduction of the gauge group has been shown by Bianchi, Pradisi and Sagnotti [11] for the type I superstring. Similar models for the heterotic string were constructed by Chaudhuri, Hockney and Lykken [13]. The 8 dimensional heterotic models of [13] were argued in [14] to correspond to topologically non-trivial $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ vector bundles, giving an example of the duality between the two heterotic string theories. Other models of [13] were interpreted in [3] and [12]. The holonomies constructed in this paper and in [5] can be used to construct 7-dimensional models where the rank of the gauge group is reduced by $4, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16$ (the reduction of the rank for an E_8 theory is $4, 6, 7, 8$; for $E_8 \times E_8$ each of the two factors can be reduced seperately). Comparing this list with the possibilities listed in the introduction of the first reference in [14], we see that they do not overlap completely, so at least some of the 7 dimensional models constructed this way are not related to the models of [13, 14]. It is somewhat surprising that the possibilities for the $E_8 \times E_8$-heterotic string appear to be much larger in number than the possibilities for the $Spin(32)/\mathbb{Z}_2$-heterotic string, in spite of the conjectured equivalence of the two theories under T-duality. This is a matter that deserves further study.

We hope that explicit expressions for the holonomies, as we have constructed, may be helpful in answering these questions.

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